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Numerical Technique for Surface Opening Displacement of an External Crack in an Infinite Elastic Space

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Abstract

This paper presents a numerical technique for determination of surface opening displacements of an external circular crack embedded in an infinite elastic space. Under the applied constant displacement at infinity, a Barenblatt – Dugdale crack whose crack tip plasticity governed by Tresca and von Mises yield criteria is numerically investigated. The proposed technique of a discretizing approach is qualitatively developed. It is then employed for the purpose of predicting the opening displacement of such crack with von Mises yielding criterion whose corresponding analytical expressions are not available. By superposition of Dugdale model's displacement functions, the influence function which is the essential basis of numerical implementation is obtained. The accuracy and convergence of the proposed technique has been judged by the comparison studies between Tresca analytic crack opening displacement and the one obtained from these numerical computations. Accordingly, this technique provides the sufficiently accurate results and shows a good agreement with the exact value and always converges to a stable solution. This leads to a conclusion that the problem can be estimated accurately by the simple numerical technique proposed within. Finally, the surface opening displacement of the cohesive external crack is compared for the three different yield conditions namely Dugdale, Tresca and von Mises.

Keywords: Crack opening displacement; Tresca yield criterion; Von Mises yield criterion; Surface opening displacement; External crack.

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1. INTRODUCTION

Crack whose vicinity occupied by plastic zone was originally introduced by (Barenblatt 1962) and also (Dugdale 1962). After these pioneer contributions were introduced, cohesive crack model has been studied extensively. There are some more approaches which can treat crack problem, such as (Sneddon 1946) and (Green and Zerna 1954). For the case of external crack, an analytical was studied by (Lowengrub and Sneddon 1965), and then (Weaver 1977). (Stallybrass, 1981a) commented that the external crack's solution strongly depends on asymptotic behavior of boundary condition at infinity and later in (Stallybrass 1981b), the external crack problems subjected to uniaxial tension and moment at the far ends were solved mathematically. Later, (Fabrikant 1985a) introduced an analysis procedure dealing with elastic modulus which varies under the power law.

The mentioned articles mostly involved with strip yield model, however, it is well known that the material yield under the non – linear condition like von Mises. The first analytical solution of external circular crack confined by Tresca yield annular ring was introduced by (Jin, et al. 2008). Regardless of the displacement fields, they have presented both analytical and numerical approaches for solving axisymmetric crack problem with a refined Barenblatt – Dugdale approach.

Even though, some similar works w ere investigated in the literatures, but problems with Tresca condition acting as a crucial boundary condition were hardly found in the analytical analysis. By means of numerical scheme as in (Jin, et al. 2008), cohesive stress distributions corresponding to Tresca and von Mises conditions can be evaluated qualitatively. Then these quantities will be used to further evaluate the surface opening displacement here. Accordingly, their early contribution provides not only a useful benchmark, but also the essential information for this study.

2. SCOPE

2.1 Objectives

The objective of this study is to develop a numerical technique for evaluating surface opening displacement of an external circular crack problem which embedded in an infinite elastic space. By using Dugdale's model, its corresponding displacement expressions have been used as the basis functions for this technique.

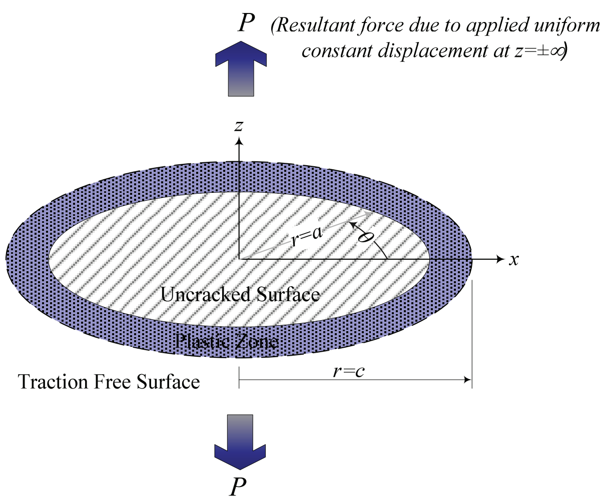

Figure 1: Geometry of an external circular crack on the boundary plane

2.2 Geometry and loading

As the proposed crack is axially symmetric, the cylindrical coordinate system (r, θ, z) is employed, as illustrated in Figure 1. A three dimensional extension of Barenblatt – Dugdale type of crack is considered, and its origin is at the center of the cross section of crack plane where $z=0$. For a circular crack of radius c under remote uniformly constant applied displacement, ε in the z – direction at infinity, a coplanar plastic yield ring is formed ahead of the crack. Cohesive zone is located on the region that $a < r < c$, and the cracked surfaces are assumed to be infinitely extended from a toward the outerbound of an infinite elastic space.

2.3 Yield criteria

Since the material in plastic zone is work – hardening, the state of stress in this region is not uniform as strip yield model assumed – to – be for mathematical simplicity. It is worth studying some other yield conditions which give the non – uniform stress distribution in plastic zone, so that Tresca and von Mises condition are investigated as yielding criteria.

3. FORMULATIONS AND INVESTIGATIONS

This section is devoted to the proposed numerical procedure being used in order to estimate the crack opening displacement and the surface opening displacement of non – linear yield criterion namely von Mises whose analytical solution is not yet found.

3.1 Computational Domain

A plastic zone is considered as the computational domain. It is formed by an annular ring extending from the tip at $r=a$ toward the separating surfaces at $r=c$. By dividing the plastic ring on the lower half space into N annular elements, the position of each discretized domain has been located by using the positive integer i and j which both vary from 1 to N. The subscript i is used to locate the loading point on the domain, which the subscript j is the position of the displacement of interest, a so – called the observed point.

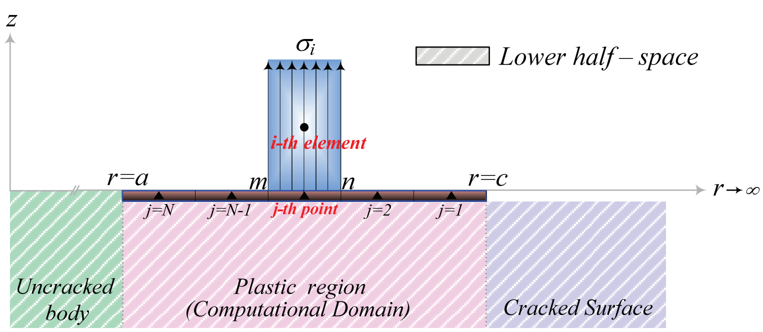

Figure 2: Annular strip loading applied on the lower half – space

Figure 2 shows an annular strip over the lower half plastic zone of a Barentblatt – Dugdale type crack on boundary plane $(z=0)$. Each of these annular strips is applied by the constant loading whose magnitude is equal to cohesive stress corresponding to the yield criterion used.

3.2 Displacement Influence Functions

The displacement influence functions representing the plastic displacement at $r=r_j$ are formulated by performing a simple superposition of Dugdale model solutions of crack in Figure 1. The loading of magnitude of, σ_i applied on the element at $r=r_i$ is assumed to be constantly distributed over its strip width, $(c-a)/N$. Then, those functions are given for all positions of the observed point defined by a distance, r_j as following expressions.

$$u_z = \frac{4(1-\nu^2)}{\pi E}\sigma_i\Big\{r_n\left[E(r_j/r_n)-E(a/r_j,r_j/r_n)\right] -r_m\left[E(r_j/r_m)-E(a/r_j,r_j/r_m)\right]\Big\}, r_j<r_i \tag{1}$$

$$u_z = \frac{4(1-\nu^2)}{\pi E}\sigma_i\Big\{r_n\left[E(r_j/r_n)-E(a/r_j,r_j/r_n)\right] -r_j\left[E(r_m/r_j)-E(a/r_m,r_m/r_j)\right] -(r_m{}^2/r_j-r_j)\left[K(r_m/r_j)-F(a/r_m,r_m/r_j)\right]\Big\}, r_j=r_i \tag{2}$$

$$u_z = \frac{4(1-\nu^2)}{\pi E}\sigma_i\Big\{(r_n^2/r_j-r_j)\left[K(r_n/r_j)-F(a/r_n,r_n/r_j)\right]+r_j\left[E(r_n/r_j)-E(a/r_n,r_m/r_j)\right] -(r_m{}^2/r_j-r_j)\left[K(r_m/r_j)-F(a/r_m,r_m/r_j)\right]-r_j\left[E(r_m/r_j)-E(a/r_m,r_m/r_j)\right]\Big\}, r_j>r_i \tag{3}$$

Where functions *F(x,y)* and *E(x,y)* denote an incomplete elliptic integral of the first and the second kind respectively. *K(y)* and *E(y)* are complete elliptic integrals of the first and the second kind.

3.3 Integrated numerical displacement

The total plastic displacement of point j can therefore be calculated by collecting the contributions of all N discretized annular loading elements. Summarizing the displacement on a specific observed point caused by N strips, plastic displacement of any j point can be given as

$$u_{zz}(r_j,0)=\frac{4(1-\nu^2)}{\pi E}\sum_{i=1}^{N}\frac{\sigma_i}{\sigma_Y}u_z(r_j,r_m,r_n) \tag{4}$$

Where σ_Y is yield strength of material. Accordingly, the surface opening displacement of the original problem contains two parts which are elastic displacement and the plastic displacement of equation (4). The total displacement can be obtained by adding the vertical displacement of punch problem as in (Maugis 2000) onto equation (4)

$$u_T(r_j,0)=\frac{4(1-\nu^2)}{\pi E}\left\{\frac{P}{4a}\cos^{-1}(a/r_j)-\sum_{i=1}^{N}\frac{\sigma_i}{\sigma_Y}u_z(r_j,r_m,r_n)\right\} \tag{5}$$

The numerical displacement in equation (5) belongs to the original problem, and it is valid for any yielding criterion of interest. This study focuses on only Tresca and von Mises, hence, their deformations have been investigated numerically using this discretizing technique proposed here.

4. RESULTS AND DISCUSSIONS

In order to determine the accuracy and convergence of the proposed numerical technique, crack opening displacement (COD) is compared between the analytical and numerical results. Trasca's exact COD can be obtained straightforwardly by substituting the elastic solution of (Jin et. al 2008) into the displacement fields derived by (Green and Zerna 1954).

4.1 Accuracy

The accuracy of the proposed technique primarily depends on some parameters used in the procedure. Table 1 shows the exact value of COD and the numerical calculated by using 10, 50 and 100 elements for *N*.

Table 1: Crack opening displacement of Tresca yield criterion

		Tresca	*Numerical Results*				
N	***a/c***	***Exact Value***	***Span Length***	***Case (1)***	***Absolute Error***	***Case (2)***	***Absolute Error***
10	0.80	1.788088	0.02500	1.788373	0.000285	1.790715	0.002627
	0.85	1.130693	0.01765	1.130864	0.000171	1.132083	0.001390
	0.90	0.635659	0.01111	0.635741	0.000082	0.636248	0.000589
	0.95	0.265687	0.00526	0.265709	0.000022	0.26583	0.000143
	0.98	0.093727	0.00204	0.093731	0.000004	0.09375	0.000023
50	0.80	1.788088	0.00500	1.788102	0.000014	1.788321	0.000233
	0.85	1.130693	0.00353	1.130702	0.000009	1.130816	0.000122
	0.90	0.635659	0.00222	0.635663	0.000004	0.635711	0.000051
	0.95	0.265687	0.00105	0.265688	0.000001	0.265699	0.000012
	0.98	0.093727	0.00041	0.093728	0	0.093729	0.000002
100	0.80	1.788088	0.00250	1.788092	0.000004	1.78817	0.000082
	0.85	1.130693	0.00176	1.130695	0.000002	1.130736	0.000043
	0.90	0.635659	0.00111	0.635660	0.000001	0.635677	0.000018
	0.95	0.265687	0.00053	0.265687	0	0.265691	0.000004
	0.98	0.093727	0.00020	0.093727	0	0.093728	0.000001

Numerical investigation of COD is categorized into 2 cases of study. The first case, *case (1),* is a numerical result obtained by using exact stress distribution over the plastic domain as an input parameter. While, the second case, *case (2),* uses the numerical procedure postulated in (Jin, et al. 2008) and further investigates for COD numerically.

It can be seen from table 1 that the most significant factor on computational accuracy turns out to be the strip length, L' which calculated by *(c-a)*/*N*. The strip length smaller than 0.001 generally gives four significant numbers for all value of *a/c*. This leads to the conclusion that as soon as the element length sufficiently small, the acceptable results is obtained, and *Case (2)* gives more accuracy as expected.

4.2 Convergence

Similar to its accuracy, the convergence of this technique depends on three parameters namely plastic zone size, *a/c* and numbers of computational elements, *N* and the strip length, L' . The convergence characteristic curves are plotted as in Figure 3. It is seen the the numerical result always converges to a stable solution as error tends toward zero as the numbers of N increases or the length, L' decreases. Thus, the convergence of this technique is quite satisfactory. As the relative errors are decreased exponentially toward the exact value, and small enough, considerably less than $1x10^{-4}$, when N = 100 is assigned.

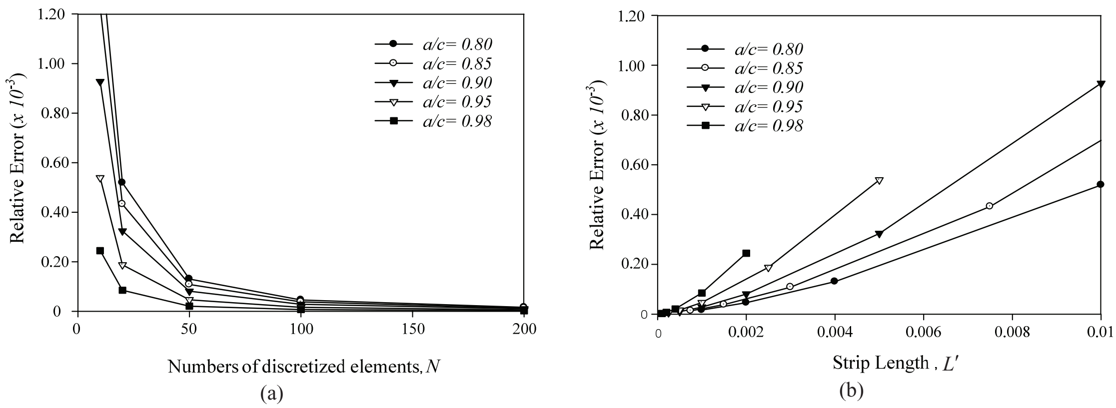

Figure 3: Relative error of COD for Poisson ratio=0.25 of various plastic zone size

In this research work, numbers of strip elements used for discretizing a computational domain also has dramatically effected on accuracy obtained. As seen in Figure 3(b), the smaller strip length, L' always produces the lesser error. Especially, when the strip length is smaller than 0.002, the relative error is mostly lower than $1x10^{-4}$.

4.4 Prediction of Von Mises displacements

Generally, yield condition at crack tip may be non – linear, hence it is worth predicting COD based on von Mises criterion. Accordingly, the comparison of three yield condition namely, Dugdale, Tresca and von Mises is employed in the case of Poisson ratio=0.25, a/c=0.80 – 0.98, N =100 and the results are tabulated in Table 2.

Table 2: Crack opening displacement of three yielding criteria

Poisson Ratio	*a/c*	*Dugdale*	*Tresca*		*Von Mises*
			Analytical	*Numerical*	
0.25	0.80	0.904185	1.788088	1.788170	2.056059
	0.85	0.605467	1.130693	1.130736	1.288776
	0.90	0.355707	0.635659	0.635677	0.713076
	0.95	0.151790	0.265687	0.265691	0.289955
	0.98	0.052683	0.093727	0.093728	0.099355
0.3	0.80	0.904185	2.235110	2.235213	2.570073
	0.85	0.605467	1.413366	1.413420	1.610970
	0.90	0.355707	0.794574	0.794596	0.891345
	0.95	0.151790	0.332109	0.332114	0.362444
	0.98	0.052683	0.117159	0.117160	0.124194
0.35	0.80	0.904185	2.980147	2.980284	3.426764
	0.85	0.605467	1.884489	1.884560	2.147960
	0.90	0.355707	1.059432	1.059462	1.188460
	0.95	0.151790	0.442811	0.442819	0.483259
	0.98	0.052683	0.156212	0.156213	0.165592

It is seen that for the same plastic zone length, von Mises criterion gives the biggest opening displacement, while Dugdale posses the smallest ones. Figure 4(a) shows the shape of surface opening of three yield conditions. There is no difference between Tresca's numerical and analytical results, and its shape located outside the plastic zone where *a/r>0.9* differs from the total plastic displacement inside plastic zone *(0.9< a/r <1.0)*.

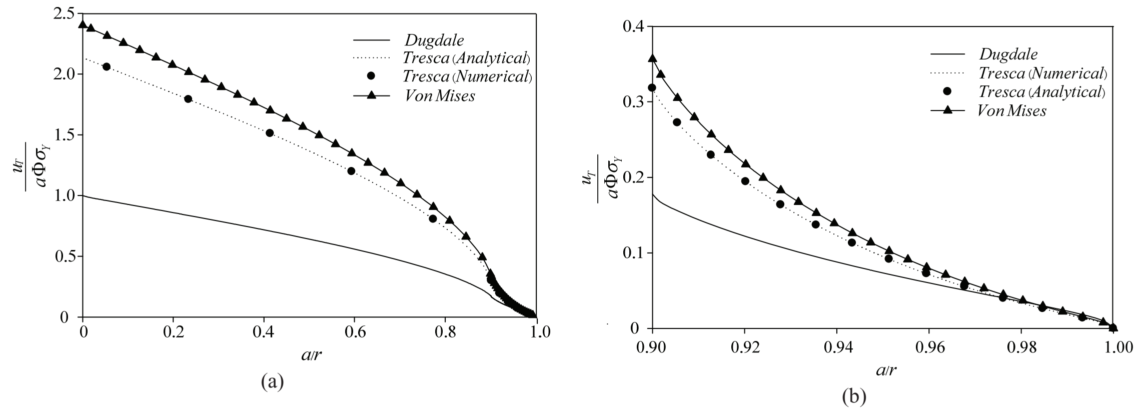

Figure 4: Prediction of the total displacement *(Poisson ratio = 0.25*, *a/c = 0.9* and *N=100)*

Figure 4(b) shows the displacement in plastic domain where crack tip plasticity occurs. The stress over yield zone trying to close the crack results in reversing deformed shape of the total plastic displacement which obviously differs from fully separated surface outside plastic zone. The point of infection on Figure 4(a) is *a/r=a/c=0.9*, or *r=c*. It is noted that the displacement in Figure 4(b), located on the left edge where *a/r=0.90*, is half the value in table 2, because it shows only surface opening displacement or half of COD value.

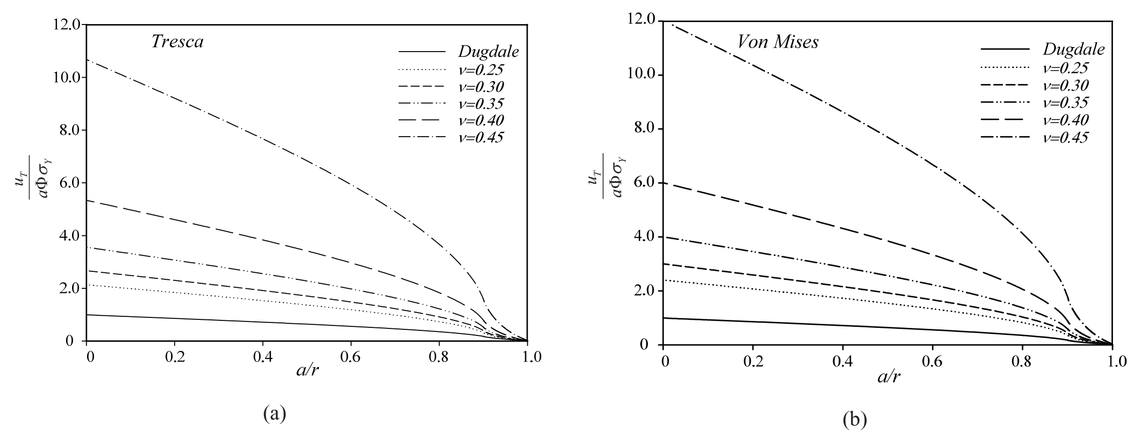

Figure 5: Surface opening displacement of various Poisson ratio

4.5 Effect of Poisson ratio

Figure 5(a) demonstrates surface opening displacement of crack under Tresca criterion and that of Dugdale model. It is obviously seen that Tresca's displacement depending on the Poisson ratio. This is in agreement with the previous work. (Keer and Mura 1965, Jin et al 2008) Similarly, the prediction of von Mises numerical results is shown in Figure 5(b). Von Mises yield condition also gives the displacements which depend on Poisson ratios Upon increasing the ratio, the surface displacement is increasing due to of the increasing of the applied stress in order to attain the same length of plastic region. Considering the same Poisson ratio, von Mises criteria gives significantly the largest displacement, because, for the same length of plastic zone, the applied stress at far fields based on von Mises condition is larger than that of Tresca criterion and Dugdale model.

5. CONCLUSIONS

The surface opening displacement of an external circular crack of Barenblatt – Dugdale type subjected to the uniformly constant displacement at far ends has been numerically investigated. For a special case of surface opening displacement, the crack opening displacement (COD) can be obtained. It has been demonstrated that the displacement of such a problem can be estimated by the simple numerical technique proposed in this study. Accordingly, the procedure has been validated by comparing its prediction of Tresca yield criterion numerical results to the corresponding analytical displacement function, and they show a reasonable agreement. It is also found that the numerical technique accuracy and convergence depend on the size of discretized element. Nevertheless, cohesive stress distribution as an input parameter has dramatically effects an accuracy of the technique for displacement calculations. Subsequently, this procedure is further applied to predict von Mises condition for both the displacement fields and COD.

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